Finiteness of homotopy groups of spheres

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Proved by Serre in [Ser53] (1953), translated in [NTG12, §2], we will follow [Hat, Thm 5.22], there are also some notes here. The statement is as follows

$$\pi_i(S^n) \cong \begin{cases} 0, & i < n \\ \mathbb{Z}, & i = n \\ \mathbb{Z} \oplus T_{i,n}, & i = 4k-1, n = 2k \\ T_{i,n}, & \text{else} \end{cases}$$

Where $T_{i,n}$ is a finite group. This is much stronger than just a rational computation of the homotopy groups it is also the statment that they are finitely generated. The $i \leq n$ cases are classical and have many elementary proofs. Recall that the (cohomological) Serre spectral sequence applies to a fibration $F \to E \to B$ over a simply connected base and is of the form $E_2^{p,q} = H^p(B; H^q(F; G)) \implies H^{p+q}(E; G)$

$n = 1$ **case.** To apply the Serre spectral sequence we will want simply connected so lets deal with the only trouble case first. Because S^1 has a contractable universal cover and [Hat02, Prop 4.1] covering space maps induce isomorphisms on homotopy groups above π_1 we see that $\pi_i(S^1) = 0$ for $i \geq 2$.

1 Reduction to rational homology

A Serre class $\mathcal{C}$ is a class of abelian groups such that: For a SES of abelian groups $A \to B \to C$ then both A and C are in $\mathcal{C}$ iff B is. There are two relevant Serre classes, that of finitely generated abelian groups and that or torsion groups. Note that these classes are closed under tensor products and tors. All homology in this section is in $\mathbb{Z}$ coefficients.

Lemma. *Consider a fibration $F \to E \to B$ over a simply connected base. Then if two out of three of the spaces have homology groups $H_n \in \mathcal{C}$ for all $n > 0$, then so does the third.*

Proof. There are three cases to consider that more or less work the same, we will just sketch it. If E is in the pair then you know that the E_∞ page of the SSS is in $\mathcal{C}$. One sees that the E_2 page is constructured from the E_∞ page out of groups extensions. The second space having homology in $\mathcal{C}$ tells you that the third space in the SES also is in $\mathcal{C}$.

The other case means that the E_2 page is completely in $\mathcal{C}$ and therefore the E_∞ is subquotients of this guy and therefore in $\mathcal{C}$, finally the homology is given by extensions of things in $\mathcal{C}$ so we are done.

□

Lemma. *If $G \in \mathcal{C}$ then $H_i K(G,n)$ is in $\mathcal{C}$ for all i, n.*

Proof. First the $n = 1$ case. We will prove it only for the specified Serre classes, not in general. First assume G is cyclic. If $G = \mathbb{Z}$ then $K(G,1) \simeq S^1$ and hence its homology is clearly finitely generated. If G is finite then apparently it is well known that the homology of $K(G,1)$ is either G or zero (depending on degree), this is proven from a cellular model called the Lens space.

In these cases the Eilenberg-Maclane spaces satisfy $K(G,1) \times K(G',1) \simeq K(G \times G',1)$ and so we can apply Kunneth to get arbitrary elements of these two Serre classes.

For the higher EM spaces there is a pathspace fibration

$$K(n-1,G) \to \mathrm{Hom}([0,1], K(n,G)) \to K(n,G)$$

where the fiber is over the base point of $K(n,G)$ and so is paths that start and end at the same place, and so we have identified $\Omega K(n,G) \simeq K(n-1,G)$. Using the $n = 1$ case and the previous lemma now implies the result.

□

Lemma. *A simply connected space, such that $\pi_i(X) \in \mathcal{C}$ for $i < n$, has a Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ whose kernel and cokernel are in $\mathcal{C}$.*

Proof. First we claim that if $\pi_i(X) \in \mathcal{C}$ for all i then so is the homology $H_i(X)$ for all $i > 0$. Recall that there is a tower of fibrations

$$\cdots \to X_n \to \cdots \to X_2 \to X_1 \to X_0$$

such that $\pi_i X_n = 0$ for $i > n$ and the fiber of the map $X_n \to X_{n-1}$ is $K(\pi_n(X), n)$. Moreover there are maps for every n, $X \to X_n$ that are $n+1$ connected (iso up to π_n). Because X is simply connected we see that $X_1 = X_0 = *$. Now it follows by induction and from the lemmas above that $H_i(X_n) \in \mathcal{C}, i > 0$, because its in a fibration with $K(G,n)$ and X_{n-1} which both have homology in $\mathcal{C}$ by induction hypothesis and Lemma. It is known that for a Postnikov system that $H_i(X) \cong H_i(X_n), i < n$, hence we can conlude the homology of X is in $\mathcal{C}$.

As we have seen the homology and homotopy groups of X in degree n are equivalent to those of X_n. Moreover so is the Hurewitz homomorphism, thus it is sufficient to prove the result for X_n. So we will look as the SSS for $K(\pi_n(X), n) \to X_n \to X_{n-1}$. Kind of lost interest, its just some algebra. You use the SSS and an induction to see that certain maps in the SSS fit in a square with the Hurwetiz homomorphism and that they must be iso mod $\mathcal{C}$ and so to therefore is the Hurweitz homomorphism.

□

Lemma. *For a simply-connected space its homotopy groups are in a Serre class $\mathcal{C}$ iff its (integral, above degree zero) homology groups are.*

Proof. From the previous lemma we see that there is a map $h : \pi_n(X) \to H_n(X)$ whose kernel is in $\mathcal{C}$ and hence a SES $\ker h \to \pi_n(X) \to \mathrm{Im} h \subseteq H_n(X)$. Since $\mathrm{Im} h$ is a subgroup of a $\mathcal{C}$ group it is a $\mathcal{C}$ group. By the property of being a Serre class we get that $\pi_n(X)$ is in $\mathcal{C}$.

For the converse apply the cokernel. Or use the Postinikov argument above.

□

2 Computation of Rational Homology

Since the sphere is a finite CW complex its homology is clearly finitely generated, so therefore is its homotopy groups. The theorem for the torsionness of the groups cant apply directly because we know that there is at least one degree that is non-finite, this will require a more subtle analysis.

2.1 Construction of a fibration.

From now on assume that $n \geq 2$. To get going with the spectral sequence we will need a fibration.

Lemma. *There is a map of space $S^n \to K(n, \mathbb{Z})$ inducing an isomorphism on π_n.*

Converting this map to a fibration we have

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & K(n,\mathbb{Z}) \\ & & \uparrow \wr \downarrow & & \\ & & S^n & & \end{array}$$

It is immediate from the LES of this fibration that $\pi_i(F) \cong \pi_i(S^n)$, for all $i \neq n, n-1$. In the interesting section we see

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \pi_n F & \xrightarrow{0} & \pi_n E & \xrightarrow{\sim} & \pi_n K(n,\mathbb{Z}) & \xrightarrow{0} & \pi_{n-1} F & \longrightarrow & 0 \\ & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \\ & & 0 & & \pi_n S^n & & \mathbb{Z} & & 0 & & \end{array}$$

where the blue zeroes are implied by the blue isomorphism that we started with. Thus $\pi_i F$ is zero for $i \leq n$ and isomorphic to $\pi_i S^n$ for $i > n$. In particular we see that computing its homotopy groups is sufficient and moreover that it does not have the $\mathbb{Z}$ in degree n; if all its homology is torsion we can apply our theorem to see all its homotopy is torsion! We want to make it the total space of a fibration to make the SSS easier, we can do this by continuing the Puppe construction we get a tower of fibrations (up to homotopy)

$$\cdots \to \Omega F \to \Omega S^n \to \Omega K(n,\mathbb{Z}) \to F \to S^n \to K(n,\mathbb{Z})$$

We now consider the SSS with $\mathbb{Q}$ coefficients of the fibration

$$K(n-1,\mathbb{Z}) = \Omega K(n,\mathbb{Z}) \to F \to S^n.$$

Recall the Cohomology. We will need to use [Hat, Prop 5.21]

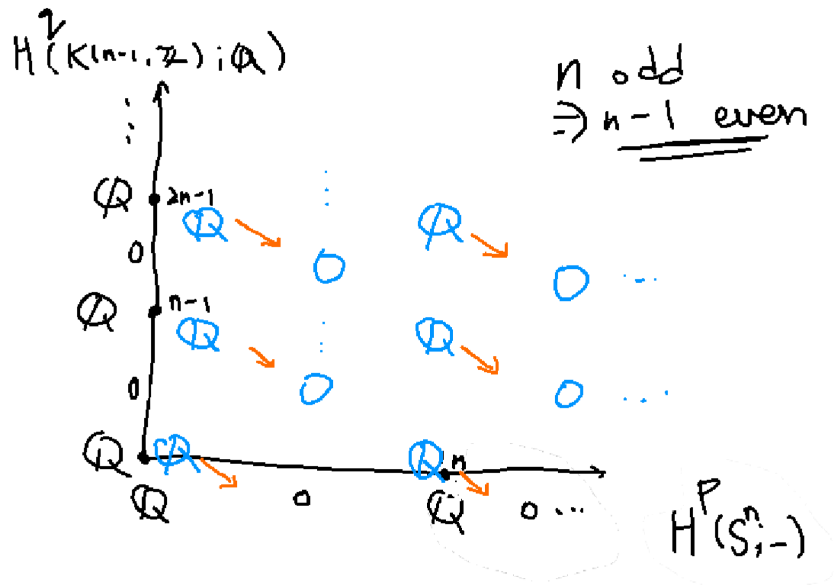
$$H^*(K(n,\mathbb{Z});\mathbb{Q}) = \begin{cases} \mathbb{Q}[x], & n \text{ even} \\ \Lambda_{\mathbb{Q}}[x], & n \text{ odd} \end{cases}$$

where $x \in H^n(K(\mathbb{Z},n);\mathbb{Q})$. Because of this, the graded peices as abelian groups are just $\mathbb{Q}$, either one in all degrees (even) or one in degree zero and one in degree n (odd). The proof is similar to that outlined by Jaden but made much simpler due to being over $\mathbb{Q}$.

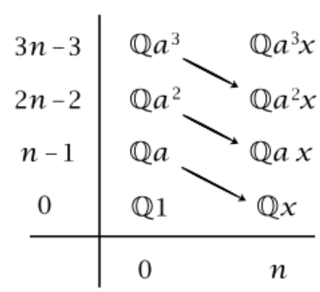
$$H^i(S^n;\mathbb{Q}) = \begin{cases} \mathbb{Q}, & i = n, 0 \\ 0, & \text{else.} \end{cases}$$

2.2 Odd Dimensional Spheres

Draw out the E_2 page of the spectral sequence



We can see immediately that the bottom left $\mathbb{Q}$ is stable. We see also that the first and only non-trivial differentials will appear on the E_n page, where $(0,q) \mapsto (n, q-n+1)$, and so we have differentials going between the $\mathbb{Q}$'s. Letting $x \in H^n(S^n;\mathbb{Z})$ be a generator, then Hatcher represents this as,



Now we begin to compute the E_∞ page. We can use that F is n connected to see that its first $H^i(F;\mathbb{Q}) = 0$ for $i \leq n$ (Hurewitz). Thus the first map is an isomorphism (really the first$n$'ish) to kill the $\mathbb{Q}$'s.

Now we induct using the product structure to see the rest are also iso. Recall that the differentials act as derivatives: For $x \in E_n^{p,q}$

$$d(xy) = (dx)y + (-1)^{p+q}x(dy).$$

The base case for $p, q = 0, n-1$ has already been done and now we induct on the q variable set as (strictly over r) $r(n-1)$. So assume that $d_n^{0,r(n-1)}$ is an isomorphism. Consider $a^{(r+1)}$ as a generator of $E_n^{0,(r+1)(n-1)}$. The differential acts as

$$d(a^{(r+1)}) = (da^r)a + (-1)^{(r+1)(n-1)}a^r(da)$$

since n is odd $n-1$ is even and we can simplify

$$d(a^{(r+1)}) = (da^r)a + \alpha a^r x$$

Using that $da = \alpha x$ $\alpha \in \mathbb{Q}_{>0}$ since the first differential is an isomorphism. Now from the induction hypothesis $da^r = \beta a^{r-1}x$ $\beta \in \mathbb{Q}_{>0}$, as its also an isomorphism. Hence

$$d(a^{(r+1)}) = \beta a^{r-1}xa + \alpha a^r x = (\alpha + \beta)a^r x.$$

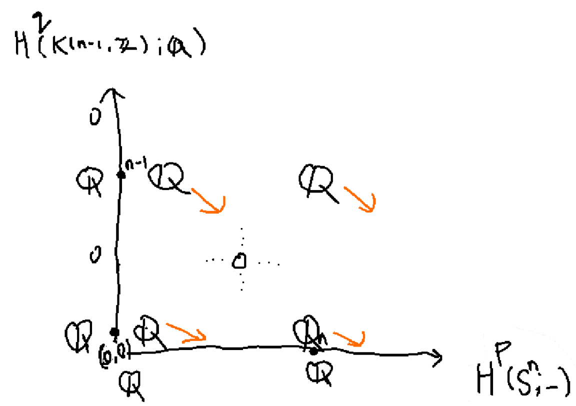
Hence it is also an isomorphism as $\alpha + \beta \in \mathbb{Q}_{>0}$.

Concluding. Thus we conclude that all $\mathbb{Q}$'s vanish on E_∞ meaning that the homology / cohomology (same over $\mathbb{Q}$) vanish for the total space of the fibration F. Hence in the odd dimensional case the homology groups are finitely generated and torsion in every degree and therefore so are the homotopy groups *of* F. Thus the homotopy groups $\pi_i S^n$ are too for $i > n$.

□

2.3 Even Dimensional Spheres

The spectral sequence gives in this case



And we can see immediatly the bottom left and top right will survive to E_∞ and by the same reason as in the odd case the first differential on E_n from top left to bottom right will be an isomorphism. Hence the space F is a rational homology sphere in dimension $2n-1$. Thus the homology is torsion up to degree $i < 2n-1$ for F and therefore so is its homotopy groups and therefore so are the homotopy groups for S^n in range $n < i < 2n-1$.

We also know that the rational homology in degree $2n-1$ of F has a $\mathbb{Q}$ and its Hurewitz homomorphism has a torsion kernel and cokernel hence the rank of its homotopy group must also be one and we can conclude it is a $\mathbb{Z}$ plus a torsion group.

Concluding. For the degrees $> 2n-1$ we have to repeat the trick we began with; construct a fibration whose total space has the homotopy groups agree only in these degrees and zero below. Then the same proof will show that they are torsion. This fibration is constructed by just attaching cells to F to kill the homotopy groups and then converting the inclusion to a fibration.

□

References

[Hat] Hatcher. Algebraic Topology Chapter 5: Spectral Sequences.

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge ; New York, 2002.

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[Ser53] Jean-Pierre Serre. Groupes D'Homotopie Et Classes De Groupes Abeliens. *The Annals of Mathematics*, 58(2):258, September 1953.